“PRICE THE OPTION”

<u>Notes and Formulas for</u>
Models of Financial Economics
Actuarial Examination
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1. PUT CALL PARITY = EQUALITY IN THE FINANCIAL MARKETS

Using the idea of a "perfect" market, the framework for the market can be defined. The key notion here is to establish some sort of equilibrium regarding the relationship of buying something vis-a-vis selling something. The idea is that if you subtract the *present value* of the strike price from the current stock price, it is the equivalent of subtracting the right to sell from the right to buy.
Put Call Parity or PCP (based on a replicating portfolio) as it is called comes in many forms to quantify the many different assets that exist. The left side is ostensibly always the same thing, Call Price – Put Price. Call is the option to buy and Put is the option to sell. An option is like a coupon with an expiration date. The right side differs though, we may obtain… **C – P =**

1. GENERAL (No dividends) = $S_0 - K \cdot e^{-rT}$
2. DISCRETE DIVIDENDS = $S_0 - K \cdot e^{-rT} - PV \sum Dividends$ where we have $\sum Dividend \cdot e^{-rt}$. Watch negative signs for these problems! Keep in mind time of purchase to determine where you are in the dividend stream.
3. CONTINUOUS DIVIDENDS = $S_0 \cdot e^{-\delta T} - K \cdot e^{-rT}$
4. BOND = $B_T - K \cdot e^{-rT} - PV \sum Coupons$ Bonds do not pay dividends BUT do pay coupons. So coupons replace dividends in the PCP equation for bonds and where S is replaced by B. There are two possible PV formulas for coupons, annual and continuously compounding. For annual, we have: $\sum PV\ of\ Coupons = Coupon\ Amount \cdot \frac{1-(1+r)^{-t}}{r}$ and for continuously compounding we have $Coupon\ Amount \cdot \frac{1e^{-rt}}{e^{r}-1}$.
5. CURRENCY = $X_0 \cdot e^{-r_f T} - K \cdot e^{-r_d T}$ where $X_0 = S_0$, r = r_d, $\delta = r_f$ where X_0 = Domestic Currency (denominated) per 1 Unit of Foreign Currency. Note that a call option in one currency is a put option in the other currency. For currency options, a unit of the foreign currency is the underlying asset. If the strike price of a call is the inverse of the strike price of a foreign denominated put, then the prices of the two options are related.
   With currency options, we might have to adjust one of the options to match the same amount of what we are swapping for and we

might have to convert the value from one currency to another or compute the answer in the "foreign" currency first. For puts, the "denomination" says what you are willing to get (K) by giving up 1 unit of foreign currency and for calls, the "denomination" says what you are willing to give up (K) to get 1 unit of foreign currency. Sometimes we have to determine value of 2nd option *based* on first option. Since there are two adjustments, we basically multiply the value of one option by X_0 and K. $C_d = X_0 \cdot K \cdot P_f(\frac{1}{x_o}, \frac{1}{k})$

6. EXCHANGE (Asset) Option = $F_T^P(A) - F_T^P(B)$ = Share of A for B Option – Share of B for A Option. [C(A,B) – P(A,B) Where C(A,B) = receive A, give up B and P(A,B) = give up A, receive B and C(A,B)=P(B,A)]. A = Underlying Asset and B = Strike or Benchmark Asset. We get A and give up B. There is duality with exchange options.

FORWARD & PREPAID FORWARD:

Both types give you the stock in the future at time T, the difference is when you pay.
Forward = Future Price = $S_t \cdot e^{rT}$
Forward with Discrete Dividends = $S_t \cdot e^{rT} - AV(Dividends)$
Forward with Continuous Dividends = $S_t \cdot e^{(r-\delta)T}$
When it says prepaid forward it means that F has been discounted. So the relationship is forward price = F = PPF · e^{rT} *and prepaid forward price = F*e^{-rT}*.*
For PCP for prepaid forward we have $e^{-rT}(F - K)$*.*
The prepaid forward price is equal to S_0*- PV(Dividends). If there are no dividends than the prepaid forward price =* S_0*.*
PPF means you lose out on dividends. PPF + PPF(DIV) = Stock Price
There is an assumption that when you own a stock you can reinvest the dividends earned such that @ time T you have $e^{\delta T}$ shares of stock.

To have stock at time T in the future you can:

1. Buy stock today
2. Enter into a forward agreement to pay F at time T to get the stock. Since you delay the time you have to pay, F = S raised to the risk free rate times the amount of time. Now a key point here

is that if the stock has dividends, a forward contract denies you those dividend payments so a forward contract requires the subtraction of the accumulated value of those dividends.

Arbitrage exists when any of the 3 inequalities (propositions below) are violated, regardless of pricing method. *When you see "arbitrage" in a problem, use PCP to find the discrepancy! Simple example of arbitrage is sell an asset priced too high and buy an asset priced too low.*
When you exploit arbitrage, you take action and replicate the opposite action, so for example try to obtain what is underpriced, but also then sell the "opposite" option + buy/sell share and "borrow" a strike. One method of analysis is the "cash flow" approach w/ PCP and replicating portfolios (Cash flow view is opposite of the more common "value" view). Under this view, a long position is negative because you are buying and therefore money is flowing away from you. A simple way to know how to exploit arbitrage when you know it exists is to move everything to the "greater than" side of the inequality. Create the inequality by pricing the option with the "incorrect" information. Once it is set up, move everything to greater than side and then do everything on the greater than side to take advantage.
If option is priced too high, sell option and replicate purchase of option and if option is priced too low, buy option and replicate sale of option.

A (K)onversion is buying a share of stock, selling a call and buying a put option (isolate K to one side of the equation). Think of it as SC-BP-BS. A reverse conversion is selling a share of stock, buying a call and selling a put option or BC-SP-SS. A conversion is replicating a t-bill or earning the risk free interest rate.
To replicate a stock using PCP, we obtain $S_0 = C - P + K \cdot e^{-rT}$
(this means buy a call, sell a put and lend $K \cdot e^{-rT}$*)*
When it says "at the money" it means that S_0 *= K.*
If we find that the synthetic stock < actual stock, then buy the synthetic stock and short the actual stock, the difference is your profit.
"Cash Flow" implies selling is positive (money coming in) and buying is negative (money going out)

COMPARING OPTIONS:

We often want to compare options. For example a European vs American Call One of the big ideas with options is how they change

value, inextricably with the stock/asset it is derived from. The "American" option offers the choice of early exercise over the "European" option, which is not usually favorable except when there is a dividend stream AND it is greater than the rate of investing the money needed to pay the strike.
The higher the strike, the less valuable it is as a call option and the more valuable it is as a put option. Correspondingly, the price you pay for call options with increasingly higher strike prices becomes ever cheaper as the price you pay for higher strike put options gets more expensive.

Propositions:
For $K_1 < K_2 < K_3$, we have:
1. C(K_1) ≥ C(K_2) ≥ C(K_3) The cheaper the call the higher the strike price
P(K_3) > P(K_2) $>$ P(K_1) Strike and put price are directly proportional
2. P(K_2) - P(K_1) ≤ $K_2 - K_1$ (If not sell $K_2, buy\ K_1$)
$C(K_1) - C(K_2) < (K_2 - K_1) \cdot e^{-rT}$ (If European, PV)
3. $\frac{C(K_1)-C(K_2)}{K_2-K_1} > \frac{C(K_2)-C(K_3)}{K_3-K_2}$ $\frac{P(K_2)-P(K_1)}{K_2-K_1} < \frac{P(K_3)-P(K_2)}{K_3-K_2}$

The lower and upper bounds for pricing options are provided by the following inequalities:

$$S \geq Call_{USA} \geq Call_{EUR} \geq \max\,[0, PV(F) - PV(K)]$$

$$K \geq Put_{USA} \geq Put_{EUR} \geq \max\,[0, PV(K) - PV(F)]$$

With disparate options, use provided info to establish parity.
As call K goes up, premium goes down. As put K goes up, premium goes up.

Early Exercise:

In general, if there are no dividends then it is NEVER optimal to early exercise and an American option is ostensibly a European option.

American Calls Only: If there are dividends then perhaps right before the dividends are paid. $PV(Dividends) \geq K - PV(K)$. This means the present value of the dividends are worth more than the PV of the interest on the strike, so you want to exercise. Also, not exercising a call gives you an implicit put, and in the special case where volatility is zero, it is optimal to exercise whenever $S > \frac{rK}{\delta}$

American Puts Only: Perhaps, if the PV of the interest on the strike is greater than the PV of the Dividends. $P \leq K - S$. Dividends that can be received on the underlying stock, interest on the strike and the implicit call (protection against an increase in the value of the stock) in the America put. Also, where volatility is zero, it is optimal to exercise early whenever $S < \frac{rK}{\delta}$.

ALSO: The greater the value of t (time), then the greater the value of the option. (Exception: certain dividend paying stocks). Greater the volatility, more opportunity to gain and closer to expiration, less opportunity to gain. The Strike (K) grows with the interest rate.

INVESTMENT STRATEGIES:

When you **buy a bond**, you are lending money at the risk free rate so buying a risk free investment is the same as lending @ risk free rate.

Calendar Spread = Buy Call or Put with time t1 and sell Call or Put with t2. **Strangle =** Basically a poor man's straddle, high volatility. **Straddle =** Same K used buying a call and a put, it is a bet on volatility since payoffs are far from K on the + or – side. Payoff |K – S|**. Collar =** A strategy that "collars" the loss and gain so useful in conservative strategies. **Bull Spread** = Like a call option with an upper bound, it uses two calls or 2 puts. You buy low strike and sell a high strike call (or put). **Bear Spread** = Opposite of Bull, buy higher strike call (or put) and sell lower strike call (or put). **Butterfly Spread** = involves options with 3 different strike prices, you buy K1, sell two K2 and buy K3. Strike of middle option is weighted average of other two options.

1+i=e^r This is the risk free rate which is used as a balance to determine relativity. If they tell you that the annual rate is 5% (i), then we calculate r to be ln 1.05 or .04879. Treasury securities are an example of a risk free interest-bearing vehicle.

2. PRICING VIA BINOMIAL MODEL

One of the critical ways we can make sense of key relationships in financial economics is that we can *"replicate" some portfolio in some other way*; this naturally flows from the principles of equivalence. From an algebraic standpoint, it allows us to solve for unknowns given some key information. Just like with PCP, there are many instruments to replicate and then many different forms of instruments. Bottom line, there are a lot of different formulas to map all the financially engineered instruments that exist and all the relationships with these instruments. One of the key derivatives that we seek to replicate in financial economics is the **option**. The most general way to recreate the option is through a mix of stock ownership and borrowing (think bonds). The premise of the binomial approach is that the stock can either go up by some amount or down by some amount, there are only these two options for each period (h) of the binomial model. A critical component of financial economics hinges on the risk free rate. *It is a theoretical concept that allows the "perfect market" to exist*. From the risk free rate comes the risk neutral probability p*. p* *represents the probability of an increase in the stock price*.

Steps:

1. Find u,d,p*. (p* requires knowing u,d and r.)
2. Use S_0 as initial node (X_0 for currency problems) and u,d to build binomial tree with 1,2 or 3 periods
3. Based on K and type of option, determine payoff structure for your option, then find payoffs @ every node, including terminal nodes
4. Use price formula with p* (or p) to find price of option. Start at end nodes and work backward using the following to "recalculate" each node:
   $e^{-rh}[Pullback\ Value^{U} \cdot P^{*} + Pullback\ Value^{D} \cdot (1 - P^{*})]$. (Note: It is possible for the price calculated to = S_0.

Tree Strategies:

1. *When all you are given is the stock price and possible new prices of the stock, calculate u and d based on the ratios of new prices with old prices.*
2. *If you don't know K, try different values on either sides of the price and solve. If you resulting value doesn't match the condition of K being less than or greater than the price then discard and keep trying.*
3. *For large number of periods, check final node payoffs to determine if one of the options has no payoffs (value = 0) so you can use PCP to solve.*
4. *For large number of periods, we use p =* $\binom{n}{k} \cdot p^{*(n-k)} \cdot (1-p^*)^k$
5. *The continuously compounded stock return approaches normality as the number of steps increases.*
6. *An insurance policy can be viewed as an option where the payoff node is if the policy is needed (tornado insurance for example) and the 0 payoff node if the policy is not needed.*

There are a few formulas for u and d, most common are for the standard binomial tree (Forward Tree):

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}}$$
$$p^* = \frac{e^{(r-\delta)h}-d}{u-d} = \frac{1}{1+e^{\sigma\sqrt{h}}}$$

For CRR and futures contracts trees, the product of ud is 1 and $u = e^{\sigma\sqrt{h}}$ and $d = e^{-\sigma\sqrt{h}}$ and $p^* = \frac{1-d}{u-d}$

AMERICAN OPTION NOTE:

If it is an American option with dividends, it may make sense to exercise early. Therefore we use the following to determine which payoff to use at each node for American options:

Call Payoff at each node then is given by:

max($S_t - K, e^{-rh}[Pullback\ Value^U \cdot P^* + Pullback\ Value^D \cdot (1-P^*)$)].

Replication of the Price of Option:

If we construct a 1 period tree, we can replicate what the price of the option should be. We buy Δ shares of the underlying asset and lend B at the risk free rate, we determine the price/cost of the option, V.

$$V = \Delta S + B$$

To define the variables, we have S as the original stock price, Δ = purchase if + and sell if negative. B = Bond (+ is Lending at the risk free rate and – is borrowing at the risk free rate). So if B is negative we are lending a negative amount or in other words we are borrowing. Δ is the number of shares and this number can be fractional/decimal.

	$\boldsymbol{Call}$	$\boldsymbol{Put}$
Δ	$+$	$-$
$\boldsymbol{B}$	$-$	$+$

Whatever values you obtain for Δ and B to determine the synthetic price of the option is based on you buying the option so if you want to sell it to capture arbitrage you reverse the signs of Δ and B.
For a put option, Δ is negative and B is positive. This means that buying a put option is equivalent to selling stock and lending at the risk-free rate.

S_u is the stock price when it goes up from S and S_d is the stock price when it goes down from S. V_u as the payoff on the option when the stock price goes up and V_d as the payoff on the option when the stock price goes down. With a one period tree, we can find the following:

$$\Delta = Slope(m) = e^{-\delta h}\left(\frac{V_u - V_d}{S_u - S_d}\right) \text{ and } B = e^{-rh}\left(\frac{uV_d - dV_u}{u-d}\right)$$

Also for gamma we have: $\Gamma_0 = \frac{\Delta_{\boldsymbol{u}} - \Delta_{\boldsymbol{d}}}{S_u - S_d}$, where $\Delta_{\boldsymbol{u}} = e^{-\delta h}\left(\frac{C_{uu} - C_{ud}}{S_{uu} - S_{ud}}\right)$ and $\Delta_{\boldsymbol{d}} = e^{-\delta h}\left(\frac{C_{ud} - C_{dd}}{S_{ud} - S_{dd}}\right)$

Theta: $\theta(S,0) = \frac{V(Sud,2h) - V(S,0) - \Delta(S,0)\epsilon - \frac{1}{2}\Gamma(S,0)\epsilon^2}{2h}$

If your resulting value of the option differs from the actual value of the option (some problems) then you have a potential arbitrage situation.

No arbitrage implies this inequality: $\boldsymbol{d < e^{(r-\delta)h} < u}$.

There are further modifications to the above formulas to account for the mechanics of pricing currency options, future contracts etc. Here are some additional formulas to know:

The forward Price is equal to a 1 period binomial tree:
$S_0 e^{(r-\delta)h}$, which is equal to:

$$S_0 e^{(r-\delta)h} = (p*)S_u + (1-p*)S_d \text{ (Risk Free)}$$
$$S_0 e^{(\alpha-\delta)h} = (p*)S_u + (1-p*)S_d \text{ (Realistic)}$$

Another binomial model is the **Lognormal Tree (Alternative or Jarrow-Rudd).** It has:

$$u = e^{(r-\delta-.5\sigma^2)h+\sigma\sqrt{h}} \text{ and } d = e^{(r-\delta-.5\sigma^2)h-\sigma\sqrt{h}}$$

For **options on currencies,** we have the following translations:

$$S_0 \to X_0, \qquad r \to r_d, \qquad \delta \to r_f$$

and we have:

$$u = e^{(r_d-r_f)h+\sigma\sqrt{h}} \text{ and } d = e^{(r_d-r_f)h-\sigma\sqrt{h}}$$

and P* reflects the currency variable shift as:

$$P^* = \frac{e^{(r_d-r_f)h} - d}{u-d}$$

One of the reasons we use the risk neutral approach to pricing options is that in *using real probabilities we encounter problems in knowing the rate to discount expected payoffs*.
So if we denote the appropriate per-period rate of return for the option as γ, we get the similar $V = e^{-\gamma h}[(p)V_u + (1-p)V_d]$. This is equivalent to $e^{-rh}[(p*)V_u + (1-p*)V_d]$.

$$e^{-rh}[(p*)V_u + (1-p*)V_d] = e^{-\gamma h}[(p)V_u + (1-p)V_d]$$

Risk Neutral = Realistic

This means that the price of the option is the same under the True/Realistic or Risk Free probability calculation. α (return on stock), goes with p, so for realistic, we have $p = \frac{e^{(\alpha-\delta)h}-d}{u-d}$. p > p*

$$\gamma_{put} < r < \alpha < \gamma_{call}$$

$V \cdot e^{Y} = S\Delta \cdot e^{\alpha} + B \cdot e^{r}$, where Y = annual return for option.

<u>UTILITY VALUES:</u>

Present Value of the Utility of 1 is $U_H\ or\ U_D$ (Upstate or downstate)

U = Up = H = High

D = Down = L = Low

$Q_H = p \cdot U_H$

$Q_L = (1-p) \cdot U_L$

$e^{-rh} = Q_H + Q_L$

V = $Q_U \cdot V_U + Q_D \cdot V_D$

Return on put < r < $\alpha <$ return on call

S = $Q_U \cdot S_U \cdot e^{\delta h} + Q_D \cdot S_D \cdot e^{\delta h}$

Price of Stock = $\frac{p \cdot C_H + (1-p) \cdot C_L}{1+\alpha} = C_H Q_H + C_L Q_L$

$$p^* = \frac{Q_U}{Q_U + Q_D}$$

Risk Free Bond:

$$Q_H + Q_L = \frac{1}{1+r}$$

3. Lognormal Distribution: Basis for Asset Prices (Proto Black Scholes)

The normal distribution comes up in financial economics in a variety of ways, for example continuously compounded returns are normally distributed. With the mean and standard deviation known, we can compute the probability of a particular value occurring with a normal distribution. The normal distribution appears in option pricing because it naturally occurs when summing random variables like gambling outcomes. Stock returns follow the normal distribution (RV can be any real number) while stock prices are lognormally distributed (RVs real numbers above zero) (Also, if ln(y) is *normally* distributed, then we know that the random variable y will be *lognormally* distributed. (products of independent lognormal RVs are lognormally distributed) If a stock price at time t is lognormally distributed, we can compute probabilities like the chance that the option will expire in the money. Straightforward lognormal probability calculations are what underpins the Black-Scholes model.
When m=mean and v squared equals variance and when X (Normal) and e^x (Lognormal), we have the notation:

$$X \sim N(m, v^2) \leftrightarrow$$
$$Y = e^x \sim LogN(m, v^2).$$

where $m = \mu t$ (time) and $v\ (standard\ deviation) = \partial\sqrt{t}$.
m in the lognormal is our equivalent r and v is our equivalent sigma.

$$\boldsymbol{m} = (\propto -\boldsymbol{\delta} -.\boldsymbol{5}\boldsymbol{\sigma}^{\boldsymbol{2}})\boldsymbol{t}$$
$$\boldsymbol{v}^{\boldsymbol{2}} = \boldsymbol{\sigma}^{\boldsymbol{2}}\boldsymbol{t}$$

$$\boldsymbol{ln}\left[\frac{\boldsymbol{S_T}}{\boldsymbol{S_t}}\right] \sim \boldsymbol{N}[\boldsymbol{m} = (\propto -\boldsymbol{\delta} -.\boldsymbol{5}\boldsymbol{\sigma}^{\boldsymbol{2}})\boldsymbol{t}\ , \boldsymbol{v}^{\boldsymbol{2}} = \boldsymbol{\sigma}^{\boldsymbol{2}}\boldsymbol{t}], \boldsymbol{T} > \boldsymbol{t}$$

The expression for a stock price that is lognormally distributed:

$$S_t = S_0 e^{(\alpha - \delta - .5\sigma^2)t + \sigma\sqrt{t}Z}$$

If you are trying to calculate S_t and you have u values, plug them into N(x) to find your Z value.

If the stock price S_t is lognormally distributed, then we can perform a variety of probability and expectation calculations. Lognormal has some weaknesses though, specifically: 1. Volatility is constant, 2. large stock movements do not occur and 3. stock returns are not correlated over time.
One standard deviation move up is equal to Z = 1 (two standard deviations is Z = 2) and one standard deviation move down is equal to Z = -1.
The probability that the stock price is less than the strike is given by:

$$\Pr[S_t < K] = N(-\hat{d}_2)$$

$$\Pr[S_t > K] = N(+\hat{d}_2)$$

where we have

$$\hat{d}_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (\alpha - \delta + .5\sigma^2)T}{\sigma\sqrt{T}}$$

$$\hat{d}_2 = \frac{\ln\left(\frac{S_t}{K}\right) + (\alpha - \delta - .5\sigma^2)T}{\sigma\sqrt{T}}$$

Also:

$$\hat{d}_1 - \hat{d}_2 = \sigma\sqrt{T}$$

R.B. IRM: the “a” value is α, δ = 0 and Short Term Interest Rate = S_0

$N(d_1)$ = 1 - $N(-d_1)$

Standard Normal = $Z_u = \frac{x-\mu}{\sigma}$ = N($\frac{\ln\left(\frac{St}{S0}\right)-m}{v}$)

Mean = $E(S_T) = S_0 \cdot e^{(\alpha-\delta)t}$

Median = $S_0 \cdot e^{(\alpha-\delta-.5\sigma^2)t}$

Median $\leq$ Mean and the Median = Mean $\cdot\ e^{-.5\sigma^2 t}$

Ln $\left(\frac{E(s)}{Median(s)}\right)$ = ln $e^{.5\sigma^2 t}$ = $.5\sigma^2 t$

<u>Confidence Interval:</u>

Upper Interval: $S_0 \cdot e^{[m+Z_u \cdot \sigma\sqrt{t}]}$

Lower Interval: $S_0 \cdot e^{[m-Z_u \cdot \sigma\sqrt{t}]}$

See above for reminders of what m and v are.

To find any given confidence interval C.I. (like 80% or 95%), we use the equation 1 – P = C.I. Solving for P, we then divide it by 2 and that is our N(x), which allows us to reverse solve for Z, which is used in the above formulas. Some common values are: 80%=1.28, 90%=1.6448, 95%=1.96 and 99% = 2.575.

<u>Expectation:</u>

$E(S_T|S_t) = S_o \cdot e^{(\alpha-\delta)t}$

$Var(S_T|S_t) = E(S_T|S_t)^2 \cdot (e^{v^2} - 1)$

$Cov(S_t, S_T) = E(\frac{S_T}{S_t}) \cdot Var(S_T|S_0)$

$PE(S_t|S_t > K) = S_t \cdot e^{(\alpha-\delta)t} \cdot N(\widehat{d_1})$

$PE(S_t|S_t < K) = S_t \cdot e^{(\alpha-\delta)t} \cdot N(-\widehat{d_1})$

E(Call Payoff) $= S_t \cdot e^{(\alpha-\delta)t} \cdot N(\widehat{d_1}) - K \cdot N(\widehat{d_2})$

E(Put Payoff) $= K \cdot N(-\widehat{d_2}) - S_t \cdot e^{(\alpha-\delta)t} \cdot N(\widehat{-d_1})$

Expectation = $N(\widehat{d_1}) = \frac{PE(S_t|S_t<K)}{E(S_t)}$

Conditional Expectation = $E(S_T|S_T > K) = \frac{S_o \cdot e^{(\alpha-\delta)t} \cdot N(\widehat{d_1})}{N(\widehat{d_2})}$

$E(S_T|S_T < K) = \frac{S_o \cdot e^{(\alpha-\delta)t} \cdot N(\widehat{-d_1})}{N(\widehat{-d_2})}$

4. BLACK-SCHOLES: Primary pricing model

Just like we saw earlier, we can price an option using the notion of a binomial model, where a stock can go up or down. With the binomial model, we can have one step or many steps. *If we have an infinite number of steps in the Binomial model, the option price approaches a limiting value. This value is what we compute in the Black Scholes model.* There are six inputs to keep track of in the BS model, they are:
S, the current price of the stock
K, the strike price of the option
σ, the volatility of the stock
r, the continuously compounding risk free interest rate
T, the time to expiration
δ, the dividend yield on the stock
N(x) is a function within the BS formula that is the cumulative normal distribution, namely the probability that a number drawn will be less than x. Note that N(-x) = 1 – N(x) and if d1 and d2 are opposites (positive and negatives of same number), then we have: 1 – N(d1) = N(d2).This is the component of the formula that accounts for volatility.

The BS formula also contains d_1 and d_2 terms. They have their own formulas, although one can be defined in terms of the other. See below:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Time is arbitrary in this formula as long as we remain consistent across all variables. Just like the binomial model, there are several variants to account for assets such as futures and currencies but the most common variant for Black Scholes is to calculate the price of a European call option.

Call price is given by:

$$C(S, K, \sigma, r, T, \delta) = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

Note: The higher the value of σ, the higher the call price.

The BS formula for pricing a put option follows PCP and is given by:

$$P(S,K,\sigma,r,T,\delta) = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

The **BS formula for pricing options on currencies (Garman-Kohlhagen)** replaces the dividend yield (δ) with the foreign interest rate (r_f). If the spot exchange rate is X_0 (domestic currency per unit of foreign currency) we obtain:

$$C\left(x,K,\sigma,r,T,r_f\right) = X_0 e^{-r_f T}N(d_1) - Ke^{-r_d T}N(d_2)$$

where:

$$d_1 = \frac{\ln\left(\frac{X_0}{K}\right) + \left(r - r_f + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

The **BS formula for pricing options on futures** is modified by using the futures price ($F = S_0 e^{(r-\delta)T_F}$) as the stock price and setting the dividend yield equal to the risk free rate (r). This is called the **Black formula** and is given by:

$$C(F,K,\sigma,r,T,r) = Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2)$$

and:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \left(\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Often in future problems d_2 can be written (simplified) to -d_1.

<u>Discrete Dividends:</u>

We don't use discrete dividends in Black Scholes so we back them out and replace S everywhere with $S - PV(Dividends)$ = Prepaid Forward Price.

For discrete dividends we replace S and K with F(S) and F(K) where
F(S) = S – PV(DIV)
F(K) = K*e^(-rT)

Holding Profit Problems:
Essentially the current value of your position – Original Cost plus the interest. Remember that t is the time remaining or T is time left to expiration.
Holding Profit = Current Price of Option – Original Price e^{rt}

1. Calculate option price at earliest time point.
2. Calculate option at time of closing out position (new S and t values)
3. Then use above formula

G R E E K S Δ Γ Vega θ ρ φ

A key component of financial economics is risk and computing how risk can affect pricing and in turn potential profits. One way that we are able to better assess risk exposure is through **Option Greeks**.
The Greek Portfolio is defined as $\sum_{i=1}^{n} N_i \cdot Greek_i$.

These are six tiny derivative formulas that **measure the change in the option price** when we change a single input. "Changes in the option price" is always the numerator except for Gamma. The following assumes a long position so if you are dealing with a short position all the information below is flipped, IE vega will always be negative with a short position.

They are:

+- PRICE/Delta given by Δ and really the main Greek to know formulas for etc (know characteristics of other Greeks) and which measures the option price change when the stock price increases by $1. Delta increases as the stock prices increases and is given by $\frac{change\ in\ option\ price}{change\ in\ stock\ price} = \frac{\partial V}{\partial S}$. We also have $\Delta_c = e^{-\delta T} N(d_1)$ and $\Delta_p = -e^{-\delta T} N(-d_1)$, further $0 \leq \Delta_c \leq 1\ and - 1 \leq \Delta_p \leq 0\ and\ \Delta_c - \Delta_p = e^{-\delta T}$. (if dividend yield is zero, $e^{-\delta T}\ is\ 1$)

For calls, the lower the strike price and the lower the time to expiration, the higher the delta value. Delta is also the number of shares in the replicating formula for the option price AND the elasticity formula. The more out of money the option is, the closer delta approaches zero.

++ META PRICE/Gamma given by Γ which measures the change in delta when the stock price increases by $1. $\Gamma = \frac{change\ in\ delta}{change\ in\ stock\ price} = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$. For example, if gamma is .07, then delta increases by .07 when the stock goes up $1. Also, $\Gamma_c \geq 0, \Gamma_p \geq 0,\ \ \Gamma_c = \Gamma_p$. Gamma is always positive for call and put options and is convex (happy smile) with respect to stock price. Gamma approaches zero for deep in or out of the money calls or puts.

++ VOLATILITY/Vega doesn't have a greek letter but measures the change in the option price when there is an increase in volatility of 1 percentage point. $Vega = \frac{change\ in\ option\ price}{change\ in\ volatility} = \frac{\partial V}{\partial \sigma}$. For example if volatility is .23, then it is saying that if volatility increases by 1 percentage point, then the option price will increase by .23. Also, $Vega_c \geq 0, Vega_p \geq 0, Vega_c = Vega_p$ because the volatility for a stock is the same for a call or put. Vega is always positive for call and put options.

- TIME/Theta given by θ by measures the change in the option price when there is a decrease in the time to maturity of one day. This is also known as time decay and can be thought of as the change in the option price as time advances. θ is usually negative. We have $\frac{\partial V}{\partial t}$. The more negative theta, the quicker the value declines. An exception is put options where shorter time to expiration can be worth more than those with longer time.

+- INTEREST RATE/Rho given by ρ measures the change in the option price when there is an increase in the interest rate of 1 percentage point. $\rho = \frac{change\ in\ option\ price}{change\ in\ risk\ free\ rate} = \frac{\partial V}{\partial r}$. Also $\rho_p \leq 0$, $\rho_c \geq 0$, since we have a decrease in the PV of the strike price.

-+ DIVIDEND/Psi given by φ measures the change in the option price when there is an increase in the dividend yield of 1 percentage point. $\varphi = \frac{change\ in\ option\ price}{change\ in\ dividend\ yield} = \frac{\partial V}{\partial \delta}$. Also $\varphi_c \leq 0, \varphi_p \geq 0$.

Portfolio calculations with greeks are common, when doing such problems, we assign a value of 1 when we buy and -1 when we sell. Then we multiply either 1 or -1 by the particular greek value for the particular option that we are buying or selling. We do this for each option in the portfolio and sum the products to get the greek value. Another portfolio calculation is q_n which represents the percentage of the portfolio. This is similar to 1 and -1.
It is common to discuss various spreads when dealing with Greeks, for example **Bull spread**. This is where we buy one call option with a low strike price and sell another call option with a higher strike price. A **calendar spread** involves selling a call option and buying a call option with the same strike price but a greater time to expiration. The sold option will lose its value quicker due to time decay. If we assume the strike equaled the stock price when purchased, then the most profit will occur if the stock price remains unchanged after the sold option expires.
While greeks like delta help inform us of the dollar risk of the option price relative to the stock price, **option elasticity** tells us the risk of the option relative to the stock but in percentage terms. Namely, if the stock price changes by 1%, what is the percentage change in the value of the option?
Specifically ***option elasticity*** *is defined as the percentage change in the option price divided by the percentage change in the stock price.* The percentage change in the stock price is $\frac{\epsilon}{S}$ and the percentage change in the option price is the dollar change in the option price $\epsilon\Delta$ divided by the option price V, or $\frac{\epsilon\Delta}{V}$.

Therefore **option elasticity** is $\Omega = \frac{\frac{\epsilon\Delta}{V}}{\frac{\epsilon}{S}}$ which simplifies to $\boldsymbol{\frac{S\Delta}{V}}$**.**

$$\frac{S\Delta}{V} = \Omega = \frac{\%\ \boldsymbol{change\ in\ option\ price}}{\%\ \boldsymbol{change\ in\ stock\ price}}$$

For call options, $\Omega \geq 1$ and for put options $\Omega \leq 0$. Ω decreases as the strike price K increases.

The **volatility of an option** is the elasticity times the volatility of the stock. $\sigma_{option} = \sigma_{stock} \cdot |\Omega|$.

As we saw with binomial model replication, the value of an option is the sum of some proportional shares of the stock and a position in bonds. The **"return" on the option** is thus a weighted average of the return on the stock and the risk free rate (bonds). If α is the expected rate of return on the stock, r is the risk free rate and γ is the expected return on the option we obtain $\gamma = \frac{\Delta S}{V}\alpha + \left(1 - \frac{\Delta S}{V}\right) r$.

This can be simplified to $\gamma - r = \Omega(\alpha - r)$ which translates to the **risk premium** on the option equals the risk premium on the stock times elasticity. (Use ONLY if it mentions Black Scholes in problem)

Risk Premium of Option = $\gamma - r$

Risk Premium of Stock = $\alpha - r$

$\textbf{Elasticity} = \Omega = \frac{S\Delta}{V}$

PORTFOLIO:

Value of Portfolio = Sum of the products of quantity and price. Buying is positive and selling is negative.

$$\gamma_{portfolio} - r = \Omega_{portfolio}(\alpha - r)$$

The idea of elasticity and expected return can also be extended from a single option to a *portfolio of options*, given by $\sum \# \ of \ shares \cdot \Delta$.

Elasticity of a Portfolio:

weighted average of the elasticity of its components

$$\Omega_{Portfolio} = \frac{\Delta_{Portfolio} \cdot S_0}{V_{Portfolio}}$$

$$\textbf{Elasticity of a Portfolio} = \frac{\text{Option } Price_1}{Value\ of\ Portfolio} \cdot Greek_1 + \cdots \frac{\text{Option } Price_n}{Value\ of\ Portfolio} \cdot Greek_n$$

5. Delta Hedging: Managing Risk

In order for there to be financial markets at all, there need to be market makers. They can be thought of as store owners, if you want to buy a gallon of milk or a new TV you go to a store where the store owner sells you the product at a higher price than they pay so they can have a business and you have the convenience of buying when you want to. In the financial markets, the **market makers** sell to buyers AND buy from sellers, true middlemen. Given all the volatility of the markets as well as the unpredictability of when people decide to buy or sell, market makers need to control their **risk exposure and they do this through a process called delta hedging. To _____hedge something is to set _____ = 0.** This allows them to offset the risk of an option position. Part of the way that we can understand delta hedging is the idea of **marking to market**. This is where we answer the question if I was to cash all my chips in today, what would be my overall profit or loss? Before we get to the overnight profit explanation, we will want to determine what it takes to delta hedge. It starts with the number of options being bought multiplied by the delta for that option. This result is the delta for the number of shares to buy to D.H. The key with delta hedging is to think about the perspective of the market maker as far as inflows and outflows. Once you have the initial delta hedge, you compare these numbers to the new numbers.

DETERMINE THE INVESTMENT: (to hedge where negative implies selling aka writing)
1. 0=# Bought-or-Sold*Greek + X*Greek + X2*Shares of Stock (where delta =1 and other greeks = 0)
Set the Greek equation or portfolio = 0.
2. Usually you have one equation for each Greek. Solve for the missing variable(s). Solve for gamma first.
3. Then plug values in to determine the total investment required where negative is money we receive so we lend this amount at the risk free rate.
Initial @ Day 0:
Buying is positive
Selling is negative

Determine the net of buying and selling. If the net is negative, lend the positive amount to net to zero, if the net is positive, borrow to net to zero.

If prices move significantly enough, then the market maker essentially needs to rehedge their particular position. The amount of movement in the hedged portfolio at which there is **breakeven** and thus no gain or loss to a delta hedge is: $\pm S\sigma\sqrt{h}$

Overnight Profit

The idea of my "overnight profit" is a function of change in stock prices, option prices and interest on borrowed or lent money.

Once we know our initial investment (how much buying and selling (negative investment) of options and stocks), we use the new numbers to determine the change (difference) in prices, where:
Initially selling creates a negative if the price goes up.
Initially buying creates a positive if the price goes up.
If the initial investment is negative we multiply it by the risk free rate of $e^{rt} - 1$. (for one day: $e^{\frac{r}{365}} - 1$). Sum the new totals to determine "profit".

Overnight Profit =
1. ***Gain on Options***
2. ***Gain on Stock (if negative, sell shares)***
3. ***Interest on borrowed or lent money (you earn if initial cash flow is negative)***

REBALANCING:
1. Hedging at day 0 per normal
2. @ day 1, use new delta to determine # of shares to buy or sell
3. The balance between position in options and shares you lend (+) or borrow (-).

If your position in an asset is "long", then your gain is the new price – old price of asset. In contrast, if your position in an asset is "short", then your gain is the old price – new price of asset. As an example, if you sell puts, then you are short your position in puts. The interest piece of the overnight profit calculation is found by determining the cash inflow to the market maker at time zero and then that amount can be lent out at the risk free rate of interest.

Greek Values for Stock:
Delta =1
Gamma, Vega and Rho = 0.

Now we can also use the greeks, specifically the delta and gamma to predict the new option prices so that we can determine the optimal hedging strategy to manage the inherent risk in our position.

Delta Approximation to estimate new Call Price =

$$\boldsymbol{C_{new}} = \boldsymbol{C_{old}} + \Delta(\boldsymbol{S_1} - \boldsymbol{S_0})$$

The **delta gamma approximation** for call options is given by:

$$C_{t+h} = C_t + \Delta_{\boldsymbol{t}}\epsilon + \frac{1}{2}\boldsymbol{\Gamma_t}\epsilon^2$$

For put options, the formula is the same but delta is negative so the put price will decrease if the stock price increases.
However, we can also add theta to our delta gamma approximation when a considerable amount of time occurs between the original price and when the new option price occurs. This is called the **Delta Gamma Theta approximation** and provides an even greater level of precision to our *option price prediction abilities*. The premise with the DGT is we start with an option and a particular stock price and want to express the new option price in terms of our existing information. The DGT works great for small changes of time and price. *To predict the option price*, the DGT is:

$$\boldsymbol{C_{new}} = \boldsymbol{C_{old}} + \Delta_{\boldsymbol{t}}\boldsymbol{\epsilon} + \frac{\boldsymbol{1}}{\boldsymbol{2}}\boldsymbol{\Gamma_t}\boldsymbol{\epsilon^2} + \boldsymbol{\theta_t h}$$

ϵ is the difference in stock price so a negative value if stock price goes down. H is change in time often h=1 (day). V_t is the value of the option. We can also express the change as:

$$\epsilon^2 = \sigma^2 {S_t}^2 h$$

We can also use the Black Scholes Equation when solving for the price of an option using DGT. It relates the price of a stock to the price of the option, given by:

$$(\boldsymbol{r}-\boldsymbol{\delta})\boldsymbol{S}\Delta+\mathbf{0.5}\boldsymbol{\sigma}^{\mathbf{2}}\boldsymbol{S}^{\mathbf{2}}\boldsymbol{\Gamma}+\boldsymbol{\theta}=(\boldsymbol{r})\boldsymbol{V}$$

or (if underlying asset does not pay dividends), we have:

$$rC=rS\Delta+\theta+.5\sigma^2S^2\Gamma$$

In this equation, V is equal to the value of a call (or put) option.

The **Boyle-Emanuel Formula** is a related formula for delta hedging that helps us better understand *how frequent rehedging reduces the market makers variance*. When there is frequent rehedging, it permits a better *averaging of the effects of large swings in price.* This formula can reflect the *periodic variance of return* or the *annual variance of return*. Here is the periodic variance of return:

$$Var\left[R_{h,i}\right]=\frac{1}{2}(s^2\sigma^2\Gamma h)^2$$

and annual variance:

$$Var\left[R_{h,i}\right]=\frac{1}{2}(s^2\sigma^2\Gamma)^2\text{h}$$

D.G for IRM (see IRM chapter for detail on below formulas):

$$\Delta=P_r=\ -B(t,T)\cdot P(t,T)$$

$$\Gamma=P_{rr}=B(t,T)^2\ \cdot P(t,T)$$

$$P(t,T)_{new}=P(t,T)+\Delta\epsilon+0.5\cdot\epsilon^2\cdot\Gamma$$

Market Makers Profit (SOA #69) = $-\mathbf{0.5}\boldsymbol{\Gamma}\boldsymbol{\partial}^{\mathbf{2}}-\boldsymbol{\partial h}-\boldsymbol{rh}(\boldsymbol{SA}-\boldsymbol{Call})$
= $-\boldsymbol{\theta h}-\boldsymbol{rh}(\boldsymbol{S}\Delta-\boldsymbol{Call})\ \boldsymbol{with}\ \in=\mathbf{0}$

6. EXOTIC OPTIONS: Derivative Flavors

While most of the time we work with "standard" *vanilla flavored* options, it is possible to construct any type of option with any number of conditions or characteristics. The only true limitation on how an option is defined is its marketability. If there is someone willing to buy it, then it probably exists on some scale. There are several key variants to be aware of in financial economics.

ASIAN OPTIONS (path dependent) = An option that has a payoff that is based on the *average price over some period of time*. This is path dependent because the final value of the option at expiration depends on the path of the price of the stock leading up to expiration. Asian options are always worth less and have a smaller payoff than or equal to an otherwise equivalent "normal" option because there is inherently less volatility in them.
There are technically 8 types of asian options, based on whether it is a call or put, a geometric or arithmetic average and average asset price or average strike price.
The arithmetic formula is: A(S) = $\frac{\sum_{t=1}^{n} S_t}{N}$ (add everything, divide by #)

The geometric formula is: G(S) = $(\prod_{t=1}^{N} S_t)^{\frac{1}{n}}$ (multiply everything, raise to 1/# power)

$G(S) \leq A(S)$ and as N increases value of average price option decreases and value of average strike option increases.

Use given data to find $\bar{S}$, Where $\bar{S} = \{\begin{matrix} A(S) \\ G(S) \end{matrix}$, we have:

Average *Price* Call Payoffs = max[0, $\bar{S}$ – K]
Average *Strike* Call Payoffs = max[0, S – $\bar{S}$]
Average *Price* Put Payoffs = max[0, K - $\bar{S}$]
Average *Strike* Put Payoffs = max[0, $\bar{S}$ – S]

If you encounter a binomial tree problem involving asian options:

1. Calculate binomial tree per normal

2. Use arithmetic or geometric method to find average price or average strike at end nodes (initial value not used)
3. Based on the new initial nodes, determine payoffs at each node (ud and du are different nodes btw) (Use end nodes for S)
4. Determine probability to reach each end node where uu = p*^2
5. Find the product of the probability with the payoff for each node
6. Sum all the products and discount the sum (use t, not h)

BARRIER OPTIONS (path dependent) = This is a *type of option that is payoff contingent on whether the underlying asset reaches some specific value, called the barrier*. There are three main types:
Knock-in is of the type that goes into existence if the barrier is reached, essentially becoming a "normal" option. For down and in options the price has to decline to reach barrier and up and in options the price has to increase to reach barrier.
Knock-out is the type that goes out of existence if the asset price reaches the barrier price, this can be a down-and-out or an up-and-out.
Rebate type is where some fixed payment is made when the barrier price is reached (Payment is made at time barrier is reached else they are called deferred rebate).
If $barrier < strike,$ up-and-in call = regular call.
If $barrier \geq strike,$ down-and-in put = regular put.
"Down and In" + "Down & Out" = Ordinary Option = "Up & In + Up & Out" (As long as k is consistent with above, barriers can vary.)
The starting point of S is important to determine barrier payoff! Just like asian options, barrier options are worth less than or equal to a normal option. There is a parity relationship for barrier options, it is:

Knock-in option + Knock-out option = Ordinary Option

COMPOUND OPTION (not path dependent) = *This is an option to buy an option, a sort of meta option.* By definition there are two strikes and two expirations so they are a bit more complicated than a normal option. Two events must take place in order for the compound option to have value. At the first expiration date, it must be worthwhile to exercise the right to purchase the option, and then at the second expiration, it must be worthwhile to exercise the option. The compound call option will be exercised if the underlying regular option price is worth more than the compound strike price (x). The compound put option will be exercised if the underlying regular option price is worth less than the compound strike price (x). There are four types:

<u>**Compound Call** options are:</u>

CallonCall (option to buy a call)

CallonPut (option to buy a Put)

<u>**Compound Put** options are:</u>

PutonCall (option to sell a call)

PutonPut (option to sell a put).

X is the strike price of the compound option (price of option at t),

T is the expiration of the underlying option from time 0 and

t_1 is expiration of the "compound" option aka the time to buy or sell the underlying option or not. From these four types we can establish PCP, they are:

*Call*onCall – *Put*onCall = $C_{Eur} - xe^{-rt_1}$

*Call*onPut – *Put*onPut = $P_{Eur} - xe^{-rt_1}$

The value of the underlying option at time t_1 = $V(S_{t_1}, K, T - t_1)$

The value (payoff) of a compound call at time $t_1 = \max\,[0, V\left(S_{t_1}, K, T - t_1\right) - x]$

The value of a compound put at time t_1 = $\max\,[0, x - V\left(S_{t_1}, K, T - t_1\right)]$

3 Cash flows through process:

1. Buying Compound Option
2. Buying Regular Option
3. Receiving Option Payoff (difference of option price and strike price)

GAP OPTION (not path dependent, can be negative) = *a type of option where there is one "trigger" price that determines if the option will have a non zero payoff (trigger is given by K_2) and a strike price* (*K_1*, the strike) *that determines the size of the payoff.* With gap options, <u>exercise is not optional</u> so negative payoffs and negative premiums are possible which means a gap option is not technically an "option" since you *must* exercise. If the trigger = strike, then we have a normal option. For a gap call, if the trigger is greater than the strike, there are no negative payoffs but if the trigger is less than the strike, then a negative payoff is possible. The opposite is true for a gap put, where there are

no negative payoffs if the trigger is less than the strike but a possible negative payoff if the trigger is greater than the strike price.
The payoff for a call option where the strike price K_1 determines the amount of the payoff and the trigger price K_2 determines whether the option will have a payoff, we obtain a formula where d_1 and d_2 are based on K_2 (replaces K). If trigger is hit but the payoff has not then we are in negative payoff territory.
A gap call, the stock price must be greater than trigger price K_2 to have a payoff of $S_t - K_1$ and a gap put payoff must have the stock price less than the trigger price K_2 to have payoff of $K_1 - S_t$.

$$GapCall = S_0 e^{-\delta t} N(d_1) - K_1 e^{-rt} N(d_2)$$

$$GapPut = K_1 e^{-rt} N(-d_2) - S_0 e^{-\delta t} N(-d_1)$$

PCP is thus:

$$\boldsymbol{GapCall - GapPut = S_0 e^{-\delta t} - K_1 e^{-rt}}$$

Gap options are *linearly related* to their strike price which means that gap options that are identical except for their strike price can be used to find prices of other similar gap options. Also, the more the gap option's trigger is from the strike in either direction for either call or put, the lower its value. Gap options can be built with asset or nothing and cash or nothing options.

EXCHANGE OPTION = also known as an outperformance option, is an option that only pays when the underlying asset outperforms some benchmark asset, basically where we have two assets competing. Essentially, all options are some version of an exchange option.

$$Call = F^P(A) \cdot N(d_1) - F^P(B) \cdot N(d_2)$$
$$Put = F^P(B) \cdot N(-d_2) - F^P(A) \cdot N(-d_1)$$

Call Payoff = max(0, S – K) where K is price of benchmark asset at time t.
Put Payoff = max(0, K – S) where K is price of benchmark asset at time t.

An exchange call gives the owner the right to purchase an underlying asset in exchange for a strike asset. An exchange put gives the owner the right to give up the underlying asset in exchange for the strike asset

CALL = Give up STRIKE

PUT = Give up UNDERLYING

Key is to determine which is the underlying asset and which is the strike asset. The same exchange option can be viewed as a call or put by switching the strike and underlying asset. This allows you to convert one option into another or a multiple of another by recognizing that relationship. We can also use PCP to solve for the value of an option if we know the other option (call or put) has the same underlying asset.

Since the pricing formula is a variant of the BS formula, we have a variant on the familiar d1 and d2. For $F^P(A)$ and $F^P(B)$ we discount for dividend and time. So instead of a r and dividend we have two different dividend yields, multiply each by $e^{(-\delta)T}$. These discounted amounts ($F^P(K)$ and Futures as well) are used in d1 and in regular B.S. formula.

$$d_1 = \frac{\ln\left(\frac{F^P(A)}{F^P(B)}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

and $d_2 = d_1 - \sigma\sqrt{T-t}$ and if the asset prices are equal, then $d_1 = .5\sigma$

Exchange options require a new calculation of volatility/sigma. This new formula requires a p value which is the correlation between the returns on the strike and underlying asset.

and $\boldsymbol{\sigma = \sqrt{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}}$

$\boldsymbol{\rho = Correlation\ between\ stocks}$

C(A,B) means you get A, give up B
P(A,B) means you give up A, get B
C(A,B) = P(B,A)
C(A,B) – P(A,B) = F(A) – F(B)
C(Y,2X) = P(2X, Y)
C(4X, 2Y) = 2*C(2X, Y)

If you have C(2X, Y), then F(2X)

ALL OR NOTHING OPTION = *These are options that pay (any currency) if the price is greater than the strike and nothing otherwise.* They can **pay a share of stock** (asset-or-nothing) or a $1 (cash-or-nothing CONC). Bottom line with these options, they either pay a discrete amount of something or nothing at all. These type of options are more theoretical in nature because they are easy to price but hard to hedge. There are many variations of these types of options, cash or nothing and asset or nothing (8 each).
All or nothing options aka binary options aka digital options:
Also known as cash calls, cash puts, asset calls and asset puts.
The prices of cash or nothing options can be calculated as the present value of the expected value.
If you own a cash call and a cash put, then you are guaranteed to receive $1 so the value of this portfolio is the present value of $1.
If you own an asset call and an asset put, then you are guaranteed to receive the asset so the value of this portfolio is the prepaid forward value of the asset.

The cash and asset option formulas are components of the Black Scholes formula.
This means they can be used as building blocks for other options like regular calls and puts and gap options.
For the asset or nothing and cash or nothing call we use $+d_1$ and for the asset or nothing and cash or nothing put we use $-d_1$. These can be used as building blocks for other options.
S|S>K (Asset Call), price = $S_t e^{-\delta(T-t)} \cdot N(d_1)$
S|S<K (Asset Put), price = $S_t e^{-\delta(T-t)} \cdot N(-d_1)$
1|S>K (Cash Call), price = $e^{-r(T-t)} \cdot N(d_2)$
1|S<K (Cash Put), price = $e^{-r(T-t)} \cdot N(-d_2)$

All or nothing option = AssetCall – K*CashCall.
$\sigma Call = \sigma Stock|\Omega|, \text{where } \Omega = \frac{\Delta S}{\Delta S - K \cdot CONC(K)}$
Asset Call + Asset Put = Prepaid Forward
Cash Call + Cash Put = e^{-rT}
Cash Put = -Cash Call

MAXIMUM RULES =

Max(A,B) = Max(B - A) + A
Max(A,B) = max(A – B, 0) + B
If C > 0, then Max(cA,cB) = c*Max(A,B)
If C < 0, then Max(cA,cB) = c*Min(A,B
Max(A,B) + min(A,B) = A + B
Min(A,B) - max(A,B) = A + B

~~**FORWARD START OPTION**~~ = This type of option provides the owner with an option at some specified point in the future. If we know that the strike price is some percentage of the stock price at the delivery time of the option, then we can determine the value of the forward start option at time 0.
We have the formula that gives us the value of the option at delivery time t1. The value of the forward start option at time t1 is equal to the following quantity of the underlying stock at time t1 (see formula where there is no S value and K value and the X value is the percentage of S that equals K). We then multiply this formula (value) by the present value of the stock to get the price of the forward start option.
For a call option expiring at time T whose strike is set on future date t to be S_t. STEPS: 1. Draw a time diagram. 2. Find value @ time t of Option. 3. Calculate price of Forward Start Option using prepaid forward price.
For forward start, calculate price of option using the Black Scholes framework where:
t = T - t_1
Price of the Forward Start Call Option =
$S_0 e^{-\delta t}[e^{-\delta t} \cdot N(d_1) - \ X e^{-r(T-t)} \cdot N(d_2)]$
Price of the Forward Start Option today =
$V_0 = PPF \cdot Price\ of\ Option\ using\ T - t$

$$d_1 = \frac{\ln\left(\frac{1}{X}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

~~**CHOOSER OPTION**~~ = The basic idea with a chooser option is the owner is allowed to choose at a specified time whether the option is a call or a put. The choice date is always before the expiration date. If the choice date equals the expiration date, then the chooser option is a straddle (a call and equivalent put)

At the choice date, the payoff is a max function of the payoff of the call or the payoff of the put. This max function can be rewritten several ways using algebra.
The price of the chooser option can also be expressed in two different ways
For an option that allows the owner to choose at time t whether the option will become a European call or put with strike K expiring at time T.

$$V_0 = Max(Call, Put)$$

We can then pull out the Call from the Max function to get:

$$V_0 = Call + Max(0, Put - Call)$$

$$V_0 = Call + Max(0, K - S)$$

where K – S is the Put Option @ time 0 since P – C = K - S.

@ time 0, $V_0 = C + P$ (if no dividends or interest)

With dividends or interest:
$P - C = Ke^{-rt} - S_t e^{-\delta t}$

@ time 0, $V_0 = C + e^{-\delta t} P$

7. Monte Carlo Valuation

While some options allow closed form solutions using the Black Scholes formula, others like arithmetic average price Asian options cannot be priced this way and so we used Monte Carlo valuation to price such an option.
Essentially we generate price simulations to calculate resulting option payoffs and then take the average of these payoffs and discount to present value.

We can use draws from U(0,1) to plug into the CDF calculator to determine the standard distribution value (z)

Risk neutral = r (focusing on the current price, more common)
and realistic = α (focusing on future values)

Monte Carlo Valuation is where we simulate future stock prices and then *use these simulated stock prices to compute the discounted expected payoff (or price) of the option*.

Simulating Standard Normal Variables
There are two methods for simulating standard normal variables. The first method requires 12 independent uniform random variables between 0 and 1. You then sum the 12 numbers and subtract 6 to get your Z (standard normal number). However, since you must have 12 numbers, not a very popular method. The second method is the **inversion method**. You use the provided prometric normal distribution calculator using the inverse CDF button. Type your u value into the N(x) box to get your corresponding x (Z value). This method is easier, doesn't require 12 numbers and gives you a z value for each u value so if you have 12 u values you can get 12 z values.

If we wish to draw random stock prices, we can generate a set of lognormally distributed stock prices using a set of standard normal Z values and plugging them into the **formula for simulating lognormal stock prices**, this is the PAYOFF:

$$S_T = S_o e^{(\alpha - \delta - 0.5\sigma^2)T + \sigma\sqrt{T}Z}$$

When discounting does not matter (expected payoff of a derivative), use the true distribution above, otherwise use the risk neutral distribution (expected price of a derivative), PRICE OF OPTION:

$$S_T = S_o e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z}$$

Note the only difference in the formulas is α becomes r.

To price via MCV, we compute the simulated lognormal prices given the Z values. We then multiply the stock price by these simulated numbers to get our simulated stock prices. The option payoff is the higher of 0 and the difference of our simulated prices and the strike price?(or stock price). *We then add all the option payoffs and divide by the number to determine the average option payoff. The last step is to discount the average option payoff using the risk free rate.*

$$Option\ Price = Average\ Payoff \cdot e^{-rt}$$

<u>Implied Volatility: (can back out in B.S. formula)</u>
There is no way to solve directly but we can confine the Implied volatility to particular boundaries by calculating two different option prices for two different implied volatilities. Try testing extreme values to get close to implied volatility. Calls and puts with same strike and time to expiration must have the same implied volatility. Use this if both implied and historical are provided in a Black Scholes problem.

<u>Historical Volatility:</u>
For historical volatility we will have X number of returns (usually have to compute as the ratio of the natural logarithm) and a time period of T. Then our h is total time T divided by X returns. This is an iterative summation problem but you like the calculator do the heavy lifting. You find the standard deviation of the time frame for which you are given (ie months), then annualize it by multiplying by the square root of 12. Use DATA button on calculator, find the natural log of the ratios of all the stock values in the form of $ln\frac{S_{t+1}}{S_t}$. Then hit 2^{ND}, DATA, ENTER (4 times). $\bar{x}$ is the average, $S_x = \sigma$, ($S_x\ squared = \sigma^2$). To find **annualized historical volatility** assuming monthly volatility where h = 12 we have $S_x \cdot \sqrt{h}$. We then plug the annualize historical volatility “x” into the following formula:

$$ln\frac{Last\ Value}{1st\ Value}\cdot\frac{12}{n-1}+.5x^2$$

where n is the total number of provided values. Be careful entering ratios into calculator! You can enter two columns of data and use the third column for the formula.
Daily volatility = 252 trading days

Calculator Formula for Annualized Expected Return:
$\bar{x}\cdot 12+\delta+.5\cdot(S_x\cdot\sqrt{h})^2$

Standard Deviation of M.C. estimate is S.D. divided by square root of the number of paths.

Variance Reduction Techniques:

Often time you need to find B, Covariance, Var*.

There are three main methods for reducing variance; they are the control variate method, the antithetic variate method and stratified sampling.

A method to *reduce the variance* of the simulated answer using MCV is to use the **control variate method (common exam topic)**. The idea is to estimate the error on each trial by using the price of a related option that does have a formula. One example is geometric and arithmetic Asian options which are highly correlated to one another where there is a closed form formula for geometric Asian options. Common questions are to estimate the value of the control variate and what the corresponding variance is.

Where we have:
$Y^*=Control\ variate\ estimate\ for\ Option\ Y$
$\bar{Y}=Monte\ Carlo\ estimate\ for\ Option\ Y$
$X=Exact,True\ price\ of\ Option\ X$
$\bar{X}=Monte\ Carlo\ estimate\ for\ Option\ X$
$\beta=Boyle\ Modification\ Factor\ (minimize\ variance)$
$\rho=Correlation\ Coefficient\ (rho)$

we obtain:

$$Y^* = \bar{Y} + (X - \bar{X}) \text{ where}$$
$$Var[Y^*] = Var[\bar{Y}] + Var\overline{[X]} - 2 \cdot Cov[\bar{Y}, \bar{X}]$$

Using Boyle modification, we have:

$$Y^* = \bar{Y} + \beta(X - \bar{X}) \text{ where}$$
$$\beta(X - \bar{X}) = \ Adjustment$$

$$Var[Y^*] = Var\overline{[Y]} + \beta^2 Var[\bar{X}] - 2\beta Cov\overline{[\bar{Y}, X]}$$

$Var[Y^*]$ is minimized at:

$$\beta = \frac{Cov[\bar{Y}, \bar{X}]}{Var[\bar{X}]}$$

If and only if β is set to minimize $Var[Y^*]$, we can use:

$Var[Y^*] = \overline{Var[Y]}(1 - \rho^2_{\overline{\overline{x,y}}})$ (minimal variance estimate of option y)
$Var[Y^*]$ of the control variate price for Option Y.

<u>Also:</u>

The phrase, “reduction in variance” is $Var[\bar{Y}] - Var[Y^*]$ therefore % reduction in variance $= \frac{Var[\bar{X}] - Var[X^*]}{Var[\bar{X}]}$

$$Cov[\bar{Y}, \bar{X}] = \sqrt{Var[\bar{X}]} \cdot \sqrt{Var[\bar{Y}]} \cdot \rho_{\bar{x},\bar{y}}$$

There is also the **antithetic variate (inversion) method**, which posits that for every simulation, there exists an opposite and equally likely simulation. Therefore we use the opposite of each draw to generate <u>**two** simulated outcomes for each random path we draw</u>. For every u_i, **also** simulate using $1 - u_i$ or for every z_i, also simulate using $-z_i$.
Note: Wherever you use u_i, is to plug the u_i value into the N(x) box to find your Z value, where $Z = N^{-1}(U)$.
Asian Options drawing on (0,1):

1. We make t = T/# of draws
2. Draws are in order so first draw's new stock price is used as new stock price when using next draw. (Path dependent)

There is also the **Stratified Sampling** approach, where we break the sampling space into smaller and equal size spaces and then scale the uniform numbers into the equal size spaces. We scale each uniform number by multiplying it by the size of each interval. For example, if we had 4 intervals between 0 and 1, then we multiply each number by 0.25. However, after the first number, we add where the next interval starts to the product of the uniform number and 0.25. So for example, the second number is multiplied and then .25 is added to that to get the final new number. The third number is multiplied and now 0.5 is added to that. The 4th number is multiplied by .25 and now 0.75 is added to that. However the 5th number we go back to the beginning so it is just the number times .25.

For an Asian option each stock price is used as basis for calculating next stock price (new z value) since path dependent.

When it says that a stock price follow geometric Brownian motion, it means we are using the lognormal model, so we calculate m and v.

8) BROWNIAN MOTION: Modeling w/ randomness

The idea here is that we need to find a way to model a pricing process that has *random attributes*. *Brownian motion is a stochastic process that evolves in continuous time and as such is a building block for pricing derivatives.*
This process is defined as Z(t) (pure or standard) and has the following key characteristics:
1. Z(0) = 0
2. Z(t + s) - Z(t) is normally distributed with mean 0 and variance s N(0,s).
3. Z(t) is continuous in t (a martingale means the expectation never changes, it is a process for which E[Z(t+s)|Z(t)]=Z(t).)
4. Time (increments) are independent, no Memory!
5. Quadratic variation of Brownian process (sum of the squared increments to the process) = T and cubic or higher order variation = 0.
6. Total Variation is infinite which implies the absolute length of a Brownian path is infinite over any finite interval since it moves up and down rapidly.
7. It is a diffusion process, which means the absolute value of the random variable tends to get larger. Future values do not depend on path taken to reach current value.

(Note: Z is also a standard normal random variable so notation wise need to infer context to determine if Z is in fact a stochastic random process)

A way to think about the Brownian process is *over small periods of time, changes in the value of the process are normally distributed with a variance that is proportional to the length of the time period.*

BASICS:
When a problem refers to RISK NEUTRAL, use r. When a problem refers to Realistic or Real, use α.
If you have a problem with a function $Y = e^{x+y}$, then ln y = x+ y, then d ln y is ITOS LEMMA.

1. $dX(t) = \alpha dt + \sigma dZ(t)$ ABM – Arithmetic or Standard
2. $\frac{dX(t)}{X(t)} = \alpha dt + \sigma dZ(t)$ GBM – Geometric (Ito Process)

3. $dX(t) = \lambda[\alpha - X(t)]dt + \sigma dZ(t)$ where λ is a reversion factor (speed to the mean) (O.O. Process)

These formulas come in many forms but usually the drift and volatility are variables that appear in the forms. Drift is α, the instantaneous mean per unit time, volatility is σ, and σ^2 the instantaneous variance per unit time. The key idea is you will often need to be able to go from one form to another, ie; differential, differential of associated ABM, non differential and distribution).

If you receive a question where the random variable follows Brownian motion and it asks for the probability that some value at some point in time is greater or less than some other value x, we use the normal distribution approach of: $N(\frac{x-m}{v})$.

A R I T H M E T I C

The differential form of the **arithmetic brownian motion process** is given by:

$$dX(t) = \alpha dt + \sigma\, dZ(t)$$

where α = a constant drift (can be real probability) and σ is a constant volatility term.

The NON differential form of the **arithmetic brownian motion process** is given by:

$$X(T) - X(0) = \alpha T + \sigma Z(t)$$

X(T) has a normal distribution:

$$X(T) - X(0) \sim N(\alpha t, \sigma^2 t)$$

If X(t) is modeled with a G.B.M. then there is an associated ABM given by (σ is same for both motions):

$$d[lnX(t)] = (\alpha - \frac{1}{2}\sigma^2)dt + \sigma dZ(t)$$

$$d[lnX(t)] = (\alpha - \delta - \frac{1}{2}\sigma^2)dt + \sigma dZ(t)$$

1. If given a GBM and another function in terms of GBM, find the natural log of GBM in terms of new function.
2. Use formula for $dlnX(t)$ and info from GBM to write d ln C (new function)
3. Then use the function in step 2 to write as $dC = (\alpha + \frac{1}{2}\sigma^2)Cdt + \sigma CdZ(t)$

There is however a modified form of the arithmetic brownian process that does permit **mean reversion**, it is called the **Ornstein-Uhlenbeck process.** The idea with mean reversion is that if a commodity price or interest rate is significantly high or low, it is likely to return to a value closer to the average. For example, if the interest rate is significantly high, it is likely to fall and if the value is sufficiently low, it is likely to rise. We incorporate mean reversion by modifying the drift term (λ) so it is not a constant. General Form of O.U. Process:

$$dX(t) = \lambda[\alpha - X(t)]dt + \sigma dZ(t)$$

If $\alpha = 0$, then O.U. can look like: $\frac{dX(t)}{X(t)} = -\lambda dt + \frac{\sigma}{X(t)}dZ(t)$

or $dX(t) = -\lambda X(t)dt + \sigma dZ(t)$

"Solution" of an O.O. Process =

$$X(t) = X(0) \cdot e^{-\lambda t} + \alpha(1 - e^{-\lambda t}) + \sigma \int_0^t e^{-\lambda(t-s)} \cdot dZ(s)$$

$X(t) > \alpha, X(t)\ goes\ down$
$X(t) < \alpha, X(t)\ goes\ up$
$\lambda\ goes\ up, speed\ of\ mean\ reversion\ goes\ up$.

G E O M E T R I C (good for stocks)

We can further modify the brownian motion process so that both the drift and volatility are functions of X, this is called **Geometric Brownian Motion (Type of Ito Process, and underlies Black Scholes)**.

$dX(t) = \alpha X(t)dt + \sigma X(t)dZ(t)$

and if you divide the whole formula by x(t), you get:

$$\frac{dX(t)}{X(t)} = \alpha dt + \sigma dZ(t)$$

This particular form is one of the most common approaches we use to consider the prices of stocks and other assets. Some patterns to be aware of with GBM:

The **non differential form of geometric Brownian motion** is given by (the following examples use “a” for drift and “b” for volatility):

$$X(t) = X(0)e^{\left(a-\frac{1}{2}b^2\right)t+b\cdot Z(t)}$$
$$lnX(t) = \left(a - \frac{1}{2}b^2\right)t + \sigma Z(t)$$

The **distribution of geometric Brownian motion** is given by:

$$ln\frac{X(t)}{X(0)} \sim N[m = \left(a - \frac{1}{2}b^2\right)t, v^2 = b^2 t)$$

After you find m and v, you might use (x-m)/v, where 1-N(Z> (x-m)/v). GBM is extremely popular because it is used to describe stock price movement. If stock price S_T follows GBM and has dividends then the following are equivalent:

$$dS(t) = (\boldsymbol{\alpha} - \boldsymbol{\delta})S(t)dt + \boldsymbol{\sigma}S(t)dZ(t)$$

$$\frac{dS(t)}{S(t)} = (\boldsymbol{\alpha} - \boldsymbol{\delta})dt + \boldsymbol{\sigma}dZ(t)$$

$$d\ lnS(t) = (\boldsymbol{\alpha} - \boldsymbol{\delta} - \mathbf{.5}\sigma^2)dt + \boldsymbol{\sigma}dZ(t)$$

$$S(t) = S(0)e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t+\sigma\cdot Z(t)}$$

$$ln\frac{S(t)}{S(0)} \sim N[m = (\boldsymbol{\alpha} - \boldsymbol{\delta} - \mathbf{.5}\sigma^2)t, v^2 = \boldsymbol{\sigma}^2 t)$$

Related to geometric Brownian motion is that variables are lognormally distributed. Essentially, if a variable is distributed in such a way that instantaneous percentage changes follow geometric Brownian motion, then over discrete periods of time, the variable will be lognormally distributed.

Geometric Brownian Motion Covariance:
Essentially, to calculate the covariance for a function that follows GBM, we find:

1. The second moment for small t :(e^(2*t*drift + 2*t*volatility^2/2)
2. The difference of T-t: (e^((T-t)*drift + (T-t)*volatility^2/2)
3. The first moment for small t: (e^(t*drift + t*volatility^2/2)
4. The first moment for big T: (e^(T*drift + T*volatility^2/2)

   Covariance = 1*2 – 3*4

R E L A T E D T O P I C S:

Ito's Lemma (applicable to GBM, just partial derivatives!): This is a tool for deriving the process followed by a derivative when the underlying asset follows an Ito process. It is a particular way to differentiate a stochastic process. A generalized Ito Process is given by:

$$dC(S,t) = \alpha[S,t]dt + \sigma[S,t]dZ(t)$$

ABM is an Ito Process where $\alpha[S,t] = \alpha \; and \; \sigma[S,t] = \sigma$.
GBM is an Ito Process where $\alpha[S,t] = \alpha S(t) \; and \; \sigma[S,t] = \sigma S(t)$.

When you apply Ito's Lemma, remember multiplication rules. The reasoning for **multiplication rules** with terms containing dt and dZ is that results with powers of dt greater than 1 vanish in the limit and as such:

$$dt^2 = 0$$
$$dt \times dZ = 0$$
$dZ^2 = dt$ (Remember that $dZ^2 = [dZ(t)]^2$)
$$dZ' \times dZ = \rho \cdot dt$$

Let's say we have an asset whose value V is a function of S and t and dS is given, namely by:

$$dS = h[S,t]dt + k[S,t]dZ(t)$$

So if the stock price is a function of the Brownian process Z(t), we can utilize Ito's Lemma to characterize the behavior of the stock as a function of Z(t).

$$dV = dV(S,t) = \boldsymbol{V_s} dS + \frac{1}{2}\boldsymbol{V_{ss}}(dS)^2 + \boldsymbol{V_t} dt$$

Note that the bolded parts are the first and second derivatives with respect to s and the last term is a partial derivative with respect to t. (Like the chain rule in calculus). When you are given a problem, determine what your s "is", ie x. The function that is being differentiated takes the place of V in our above example. Once you have your 3 derivatives and you plug them back into our Ito's Lemma formula, then rewrite it given known derivative functions that match what you have. Even then, you probably will need to cancel out terms to match simplified correct answer. Note that sigma never changes Unless t is explicitly provided in equation, X(t) does not really contain t and so when you differentiate there is no t term. In the $(dS)^2$, only $(dZ(t))^2 = dt$ will survive.

Using Ito's Lemma is often about differentiating V and using some algebra and plugging in what you know and using some clever substitution to solve for the missing unknown. Often it is used to find the drift term (could be zero).

If you are given a function X(t), and you know the derivative function d X(t), and then you are given a second function Y(t) in terms of X(t), then you can find d Y(T) by using Ito's Lemma to find the first and second derivatives of x and first derivative of t. The dx and dx^2 terms are just using d X(t) once and squaring it using the multiplication rules.

Differentiate C and then use algebra to plug in and rearrange
The second term in C in Ito's Lemma requires foil and everything cancels except for a dt term.

One option for solving these types of problems is the differentiated form of the natural log of the function

There is a pattern (relationship) between the derivative of the natural log of a function and the derivative of the geometric form. It is almost identical (same sigma etc) but for the dt term you add ½ times the coefficient of the dZ term squared which is usually just sigma but if sigma has a coefficient then you include that too.

If you have a problem where you have to differentiate for z, then your (dZ)^2 term in itos lemma is just dt, no need to foil the derivative function.

Another popular exam topic that can involve Brownian motion is the Sharpe Ratio. The **Sharpe ratio** is the risk premium divided by the volatility, where everything is in the same unit of time. When two assets have prices driven by the same dZ (same randomness), we can conclude they will have the same Sharpe ratio, they are "perfectly correlated". This means that the Sharpe ratio for a stock and an option on that stock are the same because they are driven by the same dZ despite the fact that they do not have the same volatility nor the same risk premium.

Where the expected return on the asset is $\propto$, r is the risk-free rate and volatility is σ, the **Sharpe ratio** is given by:

$$\phi = \frac{\propto - r}{\sigma}$$

The Sharpe ratio for a call is the same as the Sharpe Ratio for the underlying asset. The Sharpe Ratio for the put has the sign reversed.

There is also the relationship between realistic and risk neutral probabilities as follows:

Realistic/True = $\frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t)$

Risk-neutral = $\frac{dS(t)}{S(t)} = (r - \delta)dt + \sigma d\tilde{Z}(t)$

where $d\tilde{Z}(t) = dZ(t) + \phi dt$ and $\tilde{Z}(t) = Z(t) + \phi t$

E(Z(t)) = ϕt

The distribution (mean and variance) when talking about **realistic/true probability**, we have: $Z(t) \sim N(0,t)\ \alpha\ and\ in\ terms\ of\ \tilde{Z}(t) \sim N(\phi t, t)\ r$ but when talking about **risk-neutral** we have near opposite of $\tilde{Z}(t) \sim N(0,t)\ r$ and in terms of $Z(t) \sim N(-\phi t, t)\ \alpha$.

The Black Scholes Equation (popular exam topic)

This is not to be confused with the Black-Scholes Formula which is used to give the price of an option while *the Black Scholes Equation is a partial differential equation which can be used to* ***verify the price of an option*** and if it does not satisfy the equation then we have an arbitrage opportunity. It uses Ito's Lemma:

$$(r-\delta)SV_s + 0.5\sigma^2 S^2 V_{ss} + V_t = r \cdot V$$

$$(r-\delta)S\Delta + 0.5\sigma^2 S^2 \Gamma + \theta = (r - \delta *)V$$

where δ is the dividend yield on the stock and $\delta *$ is the dividend yield on the derivative. Besides the two dividend rates, we have r instead of α. This equation use for a call option is related to option greeks, where V_s becomes Δ, V_{ss}becomes Γ and V_t becomes θ.

$$(r-\delta)S\Delta + 0.5\sigma^2 S^2 \Gamma + \theta = (r - \delta *)V$$

Key to this equation is that the evolution of the price is intrinsically linked to the valuation and this equation helps describe how the asset changes from some given point. You might be given one value and then need to find the partial derivatives of that value based on the equation variables.

When you are given the price of a derivative at time t, and asked to solve for a parameter like σ, use B.S.E.

S^a

S^a we look for the expected value and forward price. It is worth nothing that when changing from a true to a risk neutral expectation, replace α with r, and vice versa and the a value = equal to the elasticity. Like with Black Scholes Equation, don't confuse δ the dividend yield on the stock with $\delta *$ the dividend yield on the derivative.
Speaking of, the δ^* (dividend yield on the derivative) formula is given by:

$$\delta^* = r - a(r - \delta) - .5a(a - 1)\sigma^2.$$

The continuously compounded return on the derivative = $\gamma = a(\alpha - r) + r$. The differential form of S^a is given by:

$$\frac{dS^a}{S^a} = [a(\alpha - \delta) + .5a(a - 1)\sigma^2]dt + a\sigma\, dZ(t)$$

and expectation by:

$$E[S(T)^a] = [S(T)^a]e^{[a(\alpha-\delta)+.5a(a-1)\sigma^2](T-t)}$$

$$F[S(T)^a] = E * [S(T)^a]$$

If forward price is provided as X, we have:

$$[(X)^a]e^{[.5a(a-1)\sigma^2](T-t)}$$

<u>How to determine Positive +σ or Negative Sigma –σ</u>
There is only one case in which we have –σ and that is when we are calculating the Sharpe Ratio. So if we are calculating $d_1, d_2, \hat{d}_1, \widehat{d_2}, S_T$ we take the absolute value of sigma σ in case it appears negative in our Ito Process.

<u>9) INTEREST RATE MODELS: PCP related to bonds</u>

These are pricing models for derivatives that depend on bond prices and interest rates. They can be referred to as interest rate derivatives There are two main types, **short-rate models** and **market models** (Black's formula). Options with payments based on interest rates can be viewed equivalently as options with payments based on bond prices. For example, an individual receives a payment when interest rates rise using a call option on the interest rate (or a put option on a bond).

<u>Vasicek Interest Rate Model</u>
This is an A.B.M O.O. Process, so it is mean reverting. r can go negative and volatility does not vary with r. Generally these problems are plug and chug.

$$dr(t) = a[b - (r)t]dt + \sigma dZ(t)$$

$a[b - (r)t]$ is therefore is the drift factor (expected change in rate).

The notation of the price is P(r,t,T). It is given by:

$$P(r,t,T) = A(t,T) \cdot e^{-B(t,T)\cdot r}$$

where $B(t,T) = \frac{1-e^{-a(T-t)}}{a}$ and A(t,T) is provided.

Other relevant formulas:

$$\phi = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

$$q[r(t),t,T] = \sigma B(t,T)$$

Yield to Maturity of Infinitely Lived Bond:

$$\bar{r} = b + \frac{\sigma\phi}{a} - \frac{\sigma^2}{2a^2}$$

$P[r(t),t,T] = e^{(-[\alpha(T-t)+\beta(T-t)r])}$ where $[\alpha(T-t) and\ \beta(T-t)]$ are constants that stay the same when difference of T-t is the same.

Cox-Ingersoll-Ross (CIR) Model
This is an A.B.M O.O. Process, so it is mean reverting (b=mean). r can go negative and volatility varies with r.

$$dr(t) = a[b - (r)]dt + \sigma\sqrt{r}dZ(t)$$

$$P(r,t,T) = A(t,T) \cdot e^{-B(t,T)\cdot r}$$

Ratio Shortcut when T-t =$T_2 - t_2$:

$$\frac{\alpha(r,t,T)}{r} = \frac{\alpha(r_2,t_2,T_2)}{r_2}$$

Black's Formula
Forward Price of Bond (F) is the ratio of the N+1 years price to the N years price. $F = \frac{P(0,T+s)}{P(0,T)}$. Some key variables to be aware of:
K = What you pay to exercise option to buy bond. (right to purchase when option expires)
The T value in d_1 is when the option expires.
$\sigma = The\ volatility\ that\ corresponds\ to\ maturity\ of\ bond$.
P(0,T) is the bond price @ time of option expiration.

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \left(\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

Price of Option on Bond = $P(0,T) \cdot [F \cdot N(d_1) - K \cdot N(d_2)]$

Black Derman Toy
1. Uses effective interest rates, not continuous
2. p* = 0.5
3. The ratio between consecutive nodes is $e^{2\sigma_t\sqrt{h}}$

These problems are often about pricing caplets and interest rate caps using the BDT discounting procedure. To discount the expected value of a payment at any node, divide it by 1 plus the interest rate at the node. For interest rate caps you need to account for payments made each time period when the interest rate exceeds the cap strike rate. Harder BDT problems involve finding yield volatilities and constructing BDT binomial trees.

Forward Rate Agreement (FRA) payoff is the difference between the forward rate at time "t + s" and Forward Interest Rate
The notation for a forward interest rate from time T to time T + s is given by:
$R_0(T, T+s)$ and expressed as:

$$R_0(T, T+s) = \frac{P(0,T)}{P(0, T+s)} - 1$$

Caplet is a call option on Forward Rate Agreement (FRA), where K_R= Strike Rate.

Payoff of FRA = $R_T(T, T+s) - R_0(T, T+s)$
Caplet Payoff = $\max\,(0, R_T(T+s) - K_R)$

Price for Floorlet = $(1+f) \cdot P(0,T) \cdot [F \cdot N(d_1) - K \cdot N(d_2]$
f is the flooring interest, F is the forward price $\frac{P(0,T+s)}{P(0,T)}$ and k is $\frac{1}{1+f}$. Use the volatility for the year of the flooring interest, and notice P(0,T) gets used twice in formula.

For interest rate trees, the initial node R_0 represents the rate from time 0 to 1 and therefore the next nodes R_u and R_d represent rates from time 1 to 2 and so on.

100 Basis Points = 1%

Binomial Short Rate Models:
To find the price of a 2 year zero coupon bond we have:

$$p^* \cdot e^{-(R_0+R_u)} + (1-p^*) \cdot e^{-(R_0+R_d)}$$

Duration Hedging (To duration hedge aT_2 year bond with a T_1 year bond. Negative results imply selling) :

$$N = \frac{(T_2 - t) \cdot P(t_1, T_2)}{(T_1 - t) \cdot (t_1, T_1)}$$

10. TIPS & TRICKS

MEDITATE & EXERCISE & COLD SHOWER & BRAIN FOODS

50% = pcp, binomial pricing, black scholes
25% delta hedging and exotic options
25% = harder content

TIPS:
The SOA likes to "couch" questions in fancy language.
Use the formulas.
Concepts are key.
TRAIN to handle 3 hours without fatigue
Concentrate on explanations of solutions.
Don't overthink.
Once in a while, you will use "trial and error" to solve.
System of equations and Quadratic formula are popular in this exam.
Try to enjoy the material.
Use the 5 questions quizzes often for weak areas
Almost everything in MFE is "continuously compounded" or e^{rt}.
Occasionally we will use "effective" or $(1+r)^t$.
If you need to find the derivative of lnX^a, we have:

$$d\ lnX^a = \frac{1}{X^a} \cdot a \cdot X = X^{-a} \cdot a \cdot X = a \cdot X^{1-a}$$

$$a^x = b$$

$$x = \frac{\ln b}{\ln a}$$

$$x(1+i)^{-1} = y$$

$$y(1+i) = x$$

$$\frac{e^{x+y}}{e^{X+z}} = \frac{e^y}{e^z} = e^{y-z}$$

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